THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Feb 14

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Part I: Additional exercises

1. Suppose $x_n \geq 0, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} (-1)^n x_n$ exists. Show that (x_n) is convergent.

   Proof: If $\lim_{n\to\infty} (-1)^n x_n := x > 0$, then from Ex 3.1.10 we know there exists a natural number M such that $(-1)^n x_n > 0$ for all $n \geq M$. However, for $2M+1 \geq M$, we have $(-1)^{2M+1} x_{2M+1} = -x_{2M+1} \leq 0$, contradiction! Similarly we also have that x cannot be less than 0.

   Therefore, $x = \lim_{n\to\infty} (-1)^n x_n = 0$. By **Theorem 3.2.9** we know

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} |(-1)^n x_n| = 0.$$

2. Let $A \subset \mathbb{R}$ be nonempty and bounded from above. By the Completeness Property, $s := \sup A$ exists in $\mathbb{R}$.

   (a) Show that there exists a sequence $(a_n) \subset A$ such that $\lim_{n\to\infty} a_n = s$.

   (b) Show that, in addition, the above convergent sequence can be chosen to be monotonically increasing (not necessarily strictly increasing).

   (c) Show that if we assume $s \notin A$ in addition, then the above convergent sequence can be taken to be **strictly** increasing. Also, given an example to illustrate that without this additional assumption, such strictly increasing sequence may not exist.

   Proof:

   (a) $s = \sup A \Rightarrow \forall n \in \mathbb{N}, \exists a_n \in A$ such that $s - \dfrac{1}{n} < a_n \leq s$. Therefore, $|a_n - s| \leq \dfrac{1}{n}, \forall n$ and thus (a_n) is a sequence in A with

$$\lim_{n\to\infty} a_n = s$$

   (b) There exists $a_1 \in A$ such that $s - 1 < a_1 \leq s$. For this a_1, there exists $a_2 \in A$ such that $s - \min(\frac{1}{2}, s - a_1) \leq a_2 \leq s$. For this a_2, there exists $a_3 \in A$ such that $s - \min(\frac{1}{3}, s - a_2) \leq a_3 \leq s$. Continue in this way we obtain a sequence $(a_n) \subset A$ which is increasing and converges to s.

   Think about why we use $\leq$ here instead of $<$.

   (c) If $s \notin A$, we have $a_1 \neq s$. Take $\varepsilon_1 = s - a_1 > 0$, there exists $a_2 \in A$ such that

$$s - \min\left(\frac{1}{2}, \varepsilon_1\right) < a_2 \Rightarrow a_1 < a_2.$$

For this $a_2 < s$, we can similarly let $\varepsilon_2 = s - a_2$ and pick $a_3 > s - \min\left(\frac{1}{3}, \varepsilon_2\right)$. Continue and we can get a **strictly increasing** sequence that converges to s.

If we do not make this additional assumption that $s \notin A$, then such strictly increasing sequence may not exist. For example, let $A = \{s\}$ be a set containing only one real number, then A is nonempty and bounded from above with $\sup A = s$. There exists an increasing sequence $(a_n) = (s, s, s, \cdots) \subset A$ which converges to s. But we cannot find such a sequence which is strictly increasing.

3. **(A generalization of Ex 3.2.15)**. Suppose $a_1, a_2, \cdots, a_k$ are k given positive real numbers. Show that

$$\lim_{n\to\infty} \left(\frac{a_1^n + a_2^n + \cdots + a_k^n}{k}\right)^{\frac{1}{n}} = \max(a_1, a_2, \cdots, a_k).$$

(Hint: apply Squeeze Theorem)

Proof: Denote $\max(a_1, a_2, \cdots, a_k)$ by $M\ (> 0)$. Then

$$\frac{M^n}{k} \leq \frac{a_1^n + a_2^n + \cdots + a_k^n}{k} \leq \frac{M^n + M^n + \cdots + M^n}{k} = M^n.$$

Because

$$\lim_{n\to\infty} (M^n)^{\frac{1}{n}} = \lim_{n\to\infty} M = M$$

and

$$\lim_{n\to\infty} \left(\frac{M^n}{k}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{M}{k^{\frac{1}{n}}} = \frac{M}{\lim\limits_{n\to\infty} k^{\frac{1}{n}}} = \frac{M}{1} = M,$$

we can conclude from Squeeze Theorem that

$$\lim_{n\to\infty} \left(\frac{a_1^n + a_2^n + \cdots + a_k^n}{k}\right)^{\frac{1}{n}} = M = \max(a_1, a_2, \cdots, a_k).$$

4. Suppose (x_n) converges to $x \in \mathbb{R}$ and $x_n \neq 0, \forall n$. Does $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ exist or not?

Solution: Case 1. If $x \neq 0$, then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{\lim\limits_{n\to\infty} x_{n+1}}{\lim\limits_{n\to\infty} x_n} = \frac{x}{x} = 1.$$

Case 2. If $x = 0$, there are more than one possibilities.

(a) Consider $(x_n) = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^3}, \cdots\right)$, then $\lim\limits_{n\to\infty} x_n = x = 0$ while $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist.

(b) You can try to construct examples showing that if $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, then the limit can be any number in $[-1, 1]$.

Part II: Some comments

1. The Monotone Convergence Theorem is a very important and powerful result. It implies the existence of the limit of a bounded monotone sequence.

   Notice that in our definition of increasing sequence, we require that $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$ where **equality is allowed to hold**. So that a sequence can be increasing and decreasing simultaneously, i.e., when all the terms are equal. And a sequence is said to be **strictly increasing (decreasing)** if $a_{n+1} > a_n$ $(a_{n+1} < a_n)$, $\forall n \in \mathbb{N}$.

2. Monotone Convergence Theorem also gives a way to compute the limit of a monotone bounded sequence by finding the supremum or infimum. However, in most cases it is not easy to evaluate this supremum (infimum) directly and we need to use other methods instead.

   In many cases, a sequence (x_n) is defined inductively, given by a recursive equation:

$$x_{n+1} = f(x_n)$$

   where $f(x)$ is a known function and the first term x_1 is also known.

   When applying MCT to theses problems, our argument usually consists of three steps (sometimes the first two steps can be combined).

   Step 1: Prove that (x_n) is bounded.

   Step 2: Prove that (x_n) is monotone.

   Step 3: Use MCT to claim that $\lim_{n\to\infty} x_n$ exists. Once we know the sequence converges, we can use the induction equation to calculate the limit of this sequence: let $n \to \infty$ and we obtain an equation of the limit x: (you may accept the last equality below for now, which is an important topic called "continuity" that we will study in later lectures)

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x).$$

   Solve this equation and then $\lim_{n\to\infty} x_n = x$ is known.

   Notice that sometimes the equation $x = f(x)$ can have more than one roots. Since the limit of a convergence sequence is unique, we need to exclude the unwanted roots. Refer to **Example 3.3.4(b)**.

   Remark: The function $f(x)$ provides almost all the information that we need to prove the monotonicity and boundedness of (x_n). A frequently used method is **Mathematical Induction** in steps 1 and 2.

Part III: other problems.

1. Use Squeeze Theorem to prove that

$$\lim_{n\to\infty} \frac{1 + 2^{\frac{1}{n}} + \cdots + n^{\frac{1}{n}}}{n} = 1.$$

   Proof: $\forall 1 \leq i \leq n$ we have

$$1 \leq i^{\frac{1}{n}} \leq n^{\frac{1}{n}}.$$

Therefore,

$$1 = \frac{1+1+\cdots+1}{n} \le \frac{1+2^{\frac{1}{n}}+\cdots+n^{\frac{1}{n}}}{n} \le \frac{n^{\frac{1}{n}}+n^{\frac{1}{n}}+\cdots+n^{\frac{1}{n}}}{n} = n^{\frac{1}{n}}.$$

It's a known result that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and thus by Squeeze Theorem we have

$$\lim_{n\to\infty} \frac{1+2^{\frac{1}{n}}+\cdots+n^{\frac{1}{n}}}{n} = 1.$$

2. Let (x_n) be a sequence of real numbers defined by

$$x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}, \forall n \in \mathbb{N}.$$

Show that (x_n) is convergent.

Proof: $\forall n, x_{n+1} = x_n + \dfrac{1}{(n+1)^2} \ge x_n \Longrightarrow (x_n)$ is increasing.

Moreover,

$$x_n \le 1 + \frac{1}{1\cdot 2} + \cdots + \frac{1}{(n-1)n} = 1 + \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2.$$

Therefore, (x_n) is bounded from above and hence also convergent.

3. (**Ex 3.3.2)** Let $x_1 > 1$ and $x_{n+1} = 2 - \dfrac{1}{x_n}, \forall n \in \mathbb{N}$. Show that (x_n) is convergent and find its limit.

Solution: Here $f(x) = 2 - \dfrac{1}{x}$.

Step 1. By mathematical induction we have that $x_n > 1$ (supplement the details yourself) and hence (x_n) is bounded from below.

Step 2. Take difference and then

$$x_{n+1} - x_n = 2 - \frac{1}{x_n} - x_n = \frac{-x_n^2 + 2x_n - 1}{x_n} = -\frac{(x_n+1)^2}{x_n}$$

Since we already have $x_n > 1$, it follows that $x_{n+1} < x_n$ for all n and thus (x_n) is decreasing.

Step 3. By MCT we know (x_n) is convergent. Let $n \to \infty$ and we have

$$x = f(x) = 2 - \frac{1}{x} \Longrightarrow x = 1.$$

Remark: If interested, you may think what will happen if $0 < x_1 < 1$.

4. **(Question 2(b) on Feb 7 continued)**. If we in addition assume that (x_n) is increasing, then the converse is true, i.e., if $\lim_{n\to\infty} A_n = x \in \mathbb{R}$, then (x_n) is convergent and

$$\lim_{n\to\infty} x_n = x.$$

Solution: By assumption we have $x_n \le x_{n+1}, \forall n$ and then

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \le \frac{x_n + x_n + \cdots + x_n}{n} = x_n. \qquad (*)$$

Now fix n and let $m > n$. Then

$$\begin{aligned} A_m &= \frac{1}{m}(x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_m) \\ &\ge \frac{x_1 + x_2 + \cdots + x_n}{m} + \frac{m-n}{m} x_n. \end{aligned}$$

Let $m \to \infty$ and we have

$$x = \lim_{m\to\infty} A_m \ge 0 + 1 \cdot x_n = x_n.$$

Combine with $(*)$ and hence

$$A_n \le x_n \le x.$$

Since $\lim_{n\to\infty} A_n = x$, by Squeeze Theorem we conclude that

$$\lim_{n\to\infty} x_n = x.$$

5. Suppose $x_1, x_2 \in \mathbb{R}$ and

$$x_{n+2} = p x_{n+1} + (1-p) x_n, \quad \forall n \in \mathbb{N}$$

where $p \in (0,1)$ is a constant. Show that (x_n) is convergent and calculate its limit.

Solution: $x_{n+2} - x_{n+1} = (p-1)(x_{n+1} - x_n) = \cdots = (p-1)^n (x_2 - x_1)$ and thus

$$\begin{aligned} x_n - x_1 &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_2 - x_1) \\ &= (p-1)^{n-2}(x_2 - x_1) + (p-1)^{n-3}(x_2 - x_1) + \cdots + (x_2 - x_1) \\ &= \frac{1 - (p-1)^{n-1}}{1 - (p-1)}(x_2 - x_1). \end{aligned}$$

Therefore,

$$x_n = x_1 + \frac{1 - (p-1)^{n-1}}{2-p}(x_2 - x_1).$$

Since $0 < p < 1$, we have $|p-1| < 1$ and then

$$\lim_{n\to\infty} x_n = x_1 + \frac{1}{2-p}(x_2 - x_1) = \frac{(1-p)x_1 + x_2}{2-p}.$$

6. **Assignment 5 Supplementary Exercise 2**: By Binomial Theorem,

$$\begin{aligned} e_n = \left(1 + \frac{1}{n}\right)^n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Fix $k \in \mathbb{N}$ and let $n > k$, then

$$e_n > 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)$$

$$+\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

Taking limits on both sides of above inequality as $n \to \infty$, we get

$$e \geq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}=x_k, \quad \forall k \in \mathbb{N}.$$

On the other hand, it is easy to check that

$$e_n \leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=x_n, \quad \forall n.$$

Therefore, $e_n \leq x_n \leq e, \quad \forall n \in \mathbb{N}$ and by Squeeze Theorem we conclude that

$$\lim_{n\to\infty} x_n = e. \qquad (**)$$

Remark: Some authors use (∗∗) as the definition of e.